MATH 141: Midterm 2

Name: key

Directions:

* Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
* **Remember to simplify each expression.**
* If you do not know how to solve a problem, try your best and/or explain in English what you would do.
* Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
		50

conceptual understanding

execution

common pitfalls

2 each

1. Find the following derivatives. You are allowed to use the Differentiation Rules.

(a) $f(x) = \pi^2$

$$f'(x) = \boxed{0}$$ since π^2 is a number.

(b) $f(x) = \underbrace{x^2}_{\text{left}} \cdot \underbrace{\sin x}_{\text{right}}$ product rule.

$$f'(x) = \sin(x) \cdot \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[\sin(x)]$$

$$= \sin(x) \cdot 2x + x^2 \cos(x) = \boxed{x \cdot (2\sin(x) + x\cos(x))}$$

(c) $f(x) = \dfrac{\sin(x^2)}{(2 - \cos x)}$ ← high ← low

Quotient rule.

inside x^2
outside $\sin(x)$

Chain.

$$f'(x) = \frac{(2-\cos(x)) \cdot \frac{d}{dx}[\sin(x^2)] - \sin(x^2) \cdot \frac{d}{dx}[2-\cos(x)]}{(2-\cos(x))^2}$$

$$= \frac{(2-\cos(x)) \cdot \cos(x^2) \cdot \frac{d}{dx}[x^2] - \sin(x^2) \cdot (-(-\sin(x)))}{(2-\cos(x))^2}$$

$$= \boxed{\frac{(2-\cos(x))\cos(x^2) \cdot 2x - \sin(x)\sin(x^2)}{(2-\cos(x))^2}}$$

outside $\sqrt{x}$
inside $\tan(x^3)$

outside $\tan(x)$
inside x^3

(d) $g(x) = \sqrt{\tan x^3}$

$$g'(x) = \frac{1}{2}\left(\tan(x^3)\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left[\tan(x^3)\right]$$

$$= \frac{1}{2\sqrt{\tan(x^3)}} \cdot \sec^2(x^3) \cdot \frac{d}{dx}\left[x^3\right]$$

$$= \boxed{\frac{3x^2\sec^2(x^3)}{2\sqrt{\tan(x^3)}}}$$

(e) $f(x) = \left(\frac{4x^2-5}{2x+1}\right)^5$ outside x^5, inside $\frac{4x^2-5}{2x+1}$

$$f'(x) = 5\left(\frac{4x^2-5}{2x+1}\right)^4 \cdot \frac{d}{dx}\left[\frac{4x^2-5}{2x+1}\right]$$ use Quotient

$$= 5\left(\frac{4x^2-5}{2x+1}\right)^4 \cdot \frac{(2x+1)\cdot\frac{d}{dx}\left[4x^2-5\right] - (4x^2-5)\cdot\frac{d}{dx}\left[2x+1\right]}{(2x+1)^2}$$

$$= 5\left(\frac{4x^2-5}{2x+1}\right)^4 \cdot \frac{(2x+1)\cdot 8x - (4x^2-5)\cdot 2}{(2x+1)^2}$$

$$= 5\left(\frac{4x^2-5}{2x+1}\right)^4 \frac{16x^2+8x-8x^2+10}{(2x+1)^2}$$

$$= 5\,\frac{(4x^2-5)^4}{(2x+1)^4} \cdot \frac{8x^2+8x+10}{(2x+1)^2}$$

$$= 5\,\frac{(4x^2-5)^4 \cdot 2 \cdot (4x^2+4x+5)}{(2x+1)^6}$$

$$= \boxed{10\,\frac{(4x^2-5)^4(4x+4x+5)}{(2x+1)^6}}$$

2. The following three equations are in implicit form. Find $\frac{dy}{dx}$.

(a) $3x^2 + 3y^4 = 4x^3 - 2y^3$

$$3 \cdot 2x + 3 \cdot 4y^3 \cdot \frac{dy}{dx} = 4 \cdot 3x^2 - 2 \cdot 3y^2 \cdot \frac{dy}{dx}$$

$$6x + \underline{12y^3 \frac{dy}{dx}} = 12x^2 \underline{- 6y^2 \frac{dy}{dx}}$$

put on one side

$$12y^3 \frac{dy}{dx} + 6y^2 \frac{dy}{dx} = 12x^2 - 6x$$

$$\frac{dy}{dx}(12y^3 + 6y^2) = 12x^2 - 6x$$

$$\frac{dy}{dx} = \frac{6x(2x-1)}{6y^2(2y+1)} = \boxed{\frac{x(2x-1)}{y^2(2y+1)}}$$

product rule.

(b) $x^3 - 4x^2y^3 + y^2 = x$

$$3x^2 - 4\left(y^3 \cdot 2x + x^2 \cdot 3y^2 \frac{dy}{dx}\right) + 2y \cdot \frac{dy}{dx} = 1$$

$$3x^2 - 8xy^3 \underline{- 12x^2y^2 \frac{dy}{dx}} \underline{+ 2y \frac{dy}{dx}} = 1$$

put on one side

$$\frac{dy}{dx}\left(2y - 12x^2y^2\right) = 1 - 3x^2 + 8xy^3$$

$$\boxed{\frac{dy}{dx} = \frac{1 - 3x^2 + 8xy^3}{2y(1 - 6x^2y)}}$$

both require chain rule

(c) $\sin(xy) = 1 - \cos(xy)$

$$\cos(xy) \cdot \frac{d}{dx}[xy] = 0 - (-\sin(xy)) \cdot \frac{d}{dx}[xy]$$

product rule on xy

$$\cos(xy) \cdot \left(y \cdot 1 + x \frac{dy}{dx}\right) = \sin(xy)\left(y \cdot 1 + x \cdot \frac{dy}{dx}\right)$$

$$y\cos(xy) + \underline{x\cos(xy)\frac{dy}{dx}} = y\sin(xy) + \underline{x\sin(xy)\frac{dy}{dx}}$$

put on same side

$$x\cos(xy)\frac{dy}{dx} - x\sin(xy)\frac{dy}{dx} = -y\cos(xy) + y\sin(xy)$$

$$\frac{dy}{dx}\left(x\cos(xy) - x\sin(xy)\right) = -y\left(\cos(xy) - \sin(xy)\right)$$

$$\frac{dy}{dx} = \frac{-y \cdot \cancel{(\cos(xy) - \sin(xy))}}{x \cdot \cancel{(\cos(xy) - \sin(xy))}}$$

$$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

3. Short answer questions:

(a) Find the most general antiderivative of $f(x) = 3x^2 - \cos x$.

$$\boxed{F(x) = x^3 - \sin(x) + C}$$

because

$$F'(x) = 3x^2 - \cos(x) + 0 = f(x)$$

(b) Suppose $f(x)$ is differentiable on (a, b). Must there exist a $c \in (a, b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

No. MVT requires

① continuous on $[a, b]$

② differentiable on (a, b)

The given assumption only implies continuous on (a, b) not $[a, b]$. So MVT assumption ① is not satisfied.

(c) If $f'(x) = g'(x)$, what is true about the relationship between $f(x)$ and $g(x)$?

$$\boxed{f(x) = g(x) + C}$$

(d) Suppose $f(x)$ is differentiable on (a, b). Must there exist both an absolute minimum and maximum on (a, b)? Why or why not?

No. EVT requires continuous on $[a, b]$.

The assumption "differentiable on (a, b)" only implies continuous on (a, b), not $[a, b]$.

4. Suppose $f(x) = \dfrac{1}{x^2 - 1}$.

(a) Find all intervals on which $f(x)$ is increasing and decreasing.

(b) Find all local minimums and maximums.

① crit #'s

$$f'(x) = \frac{(x^2-1)\cdot\frac{d}{dx}[1] - 1\cdot\frac{d}{dx}[x^2-1]}{(x^2-1)^2}$$

factor

$$= \frac{(x^2-1)\cdot 0 - 2x}{((x-1)(x+1))^2}$$

$$= -\frac{2x}{(x-1)^2(x+1)^2}$$

ⓐ solve $f'(x) = 0$

$$-\frac{2x}{(x-1)^2(x+1)^2} = 0$$

$$-2x = 0$$

$$x = 0$$

ⓑ $f'(x)$ DNE

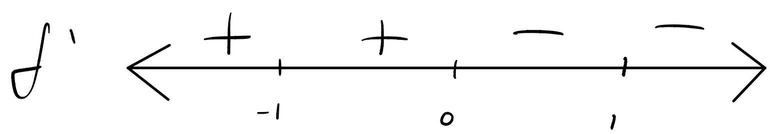
$$(x-1)^2(x+1)^2 = 0$$

$(x-1)^2 = 0$ $\qquad$ $(x+1)^2 = 0$

$x - 1 = \pm 0$ $\qquad$ $x + 1 = \pm 0$

$x = 1$ $\qquad$ $x = -1$

ⓑ Sign diagram of f'



denom is squared, always +

$x = 1$ and -1 not in domain of $f(x)$

$$f'(-2) = -\frac{2\cdot(-2)}{(-2-1)^2(-2+1)^2} = -\frac{-}{+} = +$$

$$f'(-\tfrac{1}{2}) = -\frac{2(-\frac{1}{2})}{+} = +$$

$$f'(\tfrac{1}{2}) = -\frac{2\cdot\frac{1}{2}}{+} = -$$

$$f'(2) = -\frac{2\cdot 2}{+} = -$$

∴ $f(x)$ is increasing on $(-\infty,-1)\cup(-1,0)$,

decreasing on $(0,1)\cup(1,\infty)$,

No local minimums,

local max of $f(0) = \dfrac{1}{0^2-1}$

$= -1$

(c) Now determine all intervals of concavity.

① find potential inflection points

$$f''(x) = \frac{d}{dx}\left[-\frac{2x}{(x^2-1)^2}\right]$$

chain

$$= -\frac{(x^2-1)^2 \cdot \frac{d}{dx}[2x] - 2x\,\frac{d}{dx}\left[(x^2-1)^2\right]}{\left((x^2-1)^2\right)^2}$$

$$= -\frac{(x^2-1)^2 \cdot 2 - 2x \cdot 2(x^2-1)\cdot\frac{d}{dx}[x^2-1]}{(x^2-1)^4}$$

$$= -\frac{2(x^2-1)^2 - 4x\cdot(x^2-1)\cdot 2x}{(x^2-1)^4}$$

global terms, factor out $2(x^2-1)$

$$= -\frac{2\cancel{(x^2-1)}\cdot\left((x^2-1) - 4x^2\right)}{(x^2-1)^{\cancel{4}\,3}}$$

$$= -\frac{2(-3x^2-1)}{(x^2-1)^3}$$

ⓐ $f''(x) = 0$

$$-\frac{2(-3x^2-1)}{(x^2-1)^3} = 0$$

$$-3x^2 - 1 = 0$$

$$x^2 = -\frac{1}{3}$$

$$\cancel{x = \pm\sqrt{-\tfrac{1}{3}}}$$

not real #

ⓑ $f''(x)$ DNE

$$(x^2-1)^3 = 0$$

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

② Sign diagram $f''(x)$

		-1		1	
f''	$+$		$-$		$+$

$$f''(-2) = -\frac{2\left(-3(-2)^2 - 1\right)}{\left((-2)^2-1\right)^3} = -\frac{-}{+} = +$$

$$f''(0) = -\frac{2\cdot(-3\cdot 0^2 - 1)}{(0^2-1)^3} = -\frac{-}{-} = -$$

$$f''(2) = -\frac{2(-3\cdot 2^2 - 1)}{(2^2-1)^3} = -\frac{-}{+} = +$$

$\therefore f(x)$ is concave up on $(-\infty, -1) \cup (1, \infty)$

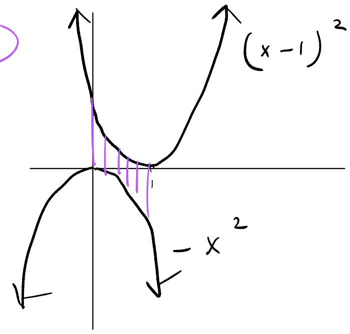
concave down on $(-1, 1)$

5. Suppose

$$f(x) = (x-1)^2 \qquad g(x) = -x^2$$

Find the minimum vertical distance between the two functions.

1



2 + 3 Minimize top − bottom or

Minimize $h(x) = f(x) - g(x)$

$$= (x-1)^2 - (-x^2)$$

$$= x^2 - 2x + 1 + x^2$$

$$= 2x^2 - 2x + 1 \quad \text{on } (-\infty, \infty)$$

4 Looking for absolute minimum on $(-\infty, \infty)$.

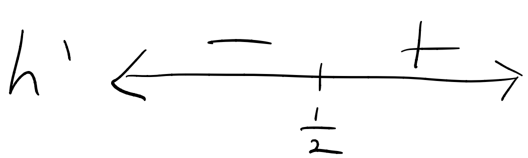
Use first derivative test.

$$h'(x) = 4x - 2$$

a $4x - 2 = 0$

$x = \frac{1}{2}$

b N/A $f'(x)$ has domain $\mathbb{R}$



$h'(0) = 4 \cdot 0 - 2 = -$

$h'(3) = 4 \cdot 3 - 2 = +$

$\therefore$ local minimum of

$$h\left(\tfrac{1}{2}\right) = 2 \cdot \left(\tfrac{1}{2}\right)^2 - 2 \cdot \tfrac{1}{2} + 1$$

$$= \frac{1}{2} - 1 + 1$$

$$= \frac{1}{2}$$

Since $h'(x) < 0$ when $x < \frac{1}{2}$ and $h'(x) > 0$ when $x > \frac{1}{2}$, $h\left(\frac{1}{2}\right) = \frac{1}{2}$ is the absolute minimum.